

# On the discrete counterparts of Cohen-Macaulay algebras with straightening laws

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## Abstract

We study properties of a poset generating a Cohen-Macaulay algebra with straightening laws (ASL for short). We show that if a poset  $P$  generates a Cohen-Macaulay ASL, then  $P$  is pure and, if  $P$  is moreover Buchsbaum, then  $P$  is Cohen-Macaulay. Some results concerning a Rees algebra of an ASL defined by a straightening closed ideal are also established. And it is shown that if  $P$  is a Cohen-Macaulay poset with unique minimal element and  $Q$  is a poset ideal of  $P$ , then  $P \uplus Q$  is also Cohen-Macaulay.

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## 1 Introduction

DeConcini, Eisenbud and Procesi defined the notion of Hodge algebra in their article [DEP] and proved many properties of Hodge algebras. They also showed that many algebras appearing in algebraic geometry and commutative ring theory have structures of Hodge algebras. In fact, the theory of Hodge algebras is an abstraction of combinatorial arguments that are used to study those rings.

A Hodge algebra is an algebra with relations which satisfy certain laws regulated by combinatorial data. It is possible to exist many Hodge algebras with the same combinatorial data. And there is the simplest Hodge algebra with given combinatorial data, called the discrete Hodge algebra. For a given Hodge algebra, we call the discrete Hodge algebra with the same combinatorial data the discrete counterpart of it.

Among the most important facts of DeConcini, Eisenbud and Procesi's results are

- A Hodge algebra and its discrete counterpart have the same dimension.
- The depth of the discrete counterpart is not greater than the depth of the original Hodge algebra.

It is known that there is a Hodge algebra whose discrete counterpart has strictly smaller depth than the original one [Hib]. And we note in Section 3 that there is a series of Cohen-Macaulay Hodge algebras of dimension  $n$  whose discrete counterparts have depth 0, where  $n$  runs over the set of all positive integers. So there is no hope to restrict the difference of the depth of a Hodge algebra and that of the discrete counterpart.

But if we restrict our attention to ordinal Hodge algebras (algebras with straightening laws, ASL for short), the influence of the combinatorial data to the ring theoretical properties become greater. So there may be a restriction to the combinatorial data by the ring theoretical properties of an ASL.

The purpose of this article is to study properties of combinatorial data of a Cohen-Macaulay graded ASL. Since it is equivalent to study the properties of combinatorial data of an ASL, i.e., the properties of the partially ordered set (poset for short) generating the ASL, and to study the properties of the discrete counterpart, our results are sometimes written in the language of posets and sometimes in the language of commutative rings.

We have to comment the result of Terai [Ter]. He asserted that if  $A$  is a homogeneous ASL, then the depth of its discrete counterpart is at least  $\text{depth } A - 1$ . But his proof also works for any graded Hodge algebras and, as is shown in Section 3, there is a Cohen-Macaulay homogeneous Hodge algebra of dimension  $n$  whose discrete counterpart is of depth 0 for any  $n \geq 1$ . So we start our research afresh.

In Section 3, we give a series of examples of Cohen-Macaulay Hodge algebras of dimension  $n$  whose discrete counterpart is of depth 0, where  $n$  runs over the set of all positive integers. It is also noted that if  $n$  is an even number, then the Hodge algebra of the example is Gorenstein.

In Section 4, we note that if a poset  $P$  generates an equidimensional ASL, then  $P$  is pure. In particular, if  $P$  generates a Cohen-Macaulay ASL, then  $P$  is pure. In Section 5, we show that if  $P$  generates a Cohen-Macaulay ASL, and  $P$  itself is Buchsbaum, then  $P$  is Cohen-Macaulay.

Consider the following four conditions.

- (i)  $P$  is a poset.
- (ii)  $P$  is a pure poset.
- (iii)  $P$  is a Buchsbaum poset.

(iv)  $P$  is a Cohen-Macaulay poset.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are well known. And the results of Sections 4 and 5 show that under the assumption that there is a Cohen-Macaulay ASL generated by  $P$ , (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii) are also valid.

Section 6 deals with some results on Rees algebras. If  $P$  is a poset and  $Q$  is a poset ideal of  $P$ , then a new poset is defined by duplicating  $Q$ . See Section 6 for the detail. We denote this poset by  $P \uplus Q$ . If  $A$  is an ASL generated by  $P$ , and  $Q$  satisfies a certain condition, the straightening closed property, then the Rees algebra of  $A$  defined by  $I = QA$  is an ASL generated by  $P \uplus Q$ .

We show that if  $P$  is Cohen-Macaulay and certain conditions on reduced Euler characteristics are satisfied, then  $P \uplus Q$  is Cohen-Macaulay. In particular, if  $P$  is a Cohen-Macaulay poset with unique minimal element, then  $P \uplus Q$  is Cohen-Macaulay for any poset ideal  $Q$ . We also show that if  $A$  is an ASL generated by  $P$ ,  $Q$  is a straightening closed poset ideal of  $P$ ,  $P$  is Cohen-Macaulay and the Rees algebra  $R$  defined by  $I = QA$  is Cohen-Macaulay, then  $P \uplus Q$ , the poset generating the ASL  $R$ , is also Cohen-Macaulay.

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## 2 Preliminaries

In this article all rings and algebras are commutative with identity. We denote the number of elements of a finite set  $X$  by  $|X|$  and, for two sets  $X$  and  $Y$ , we denote by  $X \setminus Y$  the set  $\{x \in X | x \notin Y\}$ . The set of integers (resp. non-negative integers) are denoted by  $\mathbf{Z}$  (resp.  $\mathbf{N}$ ). Standard terminology on Hodge algebras and Stanley-Reisner rings are used freely. See [DEP], [BV], [Sta, Chapter II], [BH, Chapter 5] and [Hoc] for example. However, we use the term “algebra with straightening laws” (ASL for short) to mean an ordinal Hodge algebra.

In addition we use the following notation and convention.

- We use the term poset to stand for finite partially ordered set.
- If  $P$  is a poset, we denote the set of all the minimal elements of  $P$  by  $\min P$ .
- If  $P$  is a poset, a poset ideal of  $P$  is a subset  $Q$  of  $P$  such that  $x \in Q$ ,  $y \in P$  and  $y < x$  imply  $y \in Q$ .

- For a poset  $P$ , we define the order complex  $\Delta(P)$  of  $P$  by

$$\Delta(P) := \{\sigma \subseteq P \mid \sigma \text{ is a chain}\},$$

where a chain stands for a totally ordered subset. We also define the reduced Euler characteristic  $\tilde{\chi}(P)$  of  $P$  by

$$\tilde{\chi}(P) := \tilde{\chi}(\Delta(P)).$$

- When considering a poset, we denote by  $\infty$  (or by  $-\infty$  resp.) a new element which is larger (smaller resp.) than any other element.
- If  $P$  is a poset and  $x, y \in P \cup \{\infty, -\infty\}$  with  $x < y$ , we define

$$(x, y)_P := \{z \in P \mid x < z < y\}.$$

$[x, y)_P$ ,  $(x, y]_P$  and  $[x, y]_P$  are defined similarly.

- We denote the Stanley-Reisner ring  $k[\Delta(P)]$  by  $k[P]$ , where  $k$  is a commutative ring and  $P$  is a poset. And if  $k[P]$  is Cohen-Macaulay (or Buchsbaum resp.), then we say  $P$  is Cohen-Macaulay (Buchsbaum resp.) over  $k$ .
- If  $A$  is a Hodge algebra over  $k$  generated by  $H$  governed by  $\Sigma$ , we denote by  $A_{\text{dis}}$  the discrete Hodge algebra over  $k$  generated by  $H$  governed by  $\Sigma$ .
- If  $X$  is a matrix with entries in a commutative ring  $R$ , we denote by  $I_t(X)$  the ideal of  $R$  generated by all the  $t$ -minors of  $X$ .
- If  $B$  is an  $\mathbf{N}^m$ -graded ring with  $B_{(0, \dots, 0)}$  a field, then we denote by  $\text{depth } B$  the depth of  $B_M$ , where  $M$  is the unique  $\mathbf{N}^m$ -graded maximal ideal.

Next we recall the notion of a standard subset [Miy3].

**Definition 2.1.** Let  $A$  be a Hodge algebra over  $k$  generated by  $H$  governed by  $\Sigma$ . A subset  $\Omega$  of  $H$  is called a standard subset of  $H$  if for any element  $x \in \Omega A$  and for any standard monomial  $M_i$  appearing in the standard representation

$$x = \sum_i b_i M_i \quad (0 \neq b_i \in k, M_i \text{ standard})$$

of  $x$ ,  $\text{supp } M_i$  meets  $\Omega$ .

For example, a poset ideal of  $H$  is a standard subset by Fact 2.3 below. Note that if  $\Omega$  is a standard subset of  $H$ , then  $A/\Omega A$  is a Hodge algebra over  $k$  generated by  $H \setminus \Omega$  governed by  $\Sigma/\Omega$ .

Now we recall several facts which are used in this article.

**Fact 2.2** ([DEP, Theorem 6.1 and Corollary 7.2]). *If  $A$  is a graded Hodge algebra over a field, then*

$$\dim A_{\text{dis}} = \dim A$$

and

$$\text{depth } A_{\text{dis}} \leq \text{depth } A.$$

**Fact 2.3** ([DEP, Proposition 1.2]). *If  $A$  is a Hodge algebra over  $k$  generated by a poset  $P$  governed by  $\Sigma$  and  $Q$  is a poset ideal of  $P$ , then  $A/QA$  is a Hodge algebra over  $k$  generated by  $P \setminus Q$  governed by  $\Sigma/Q$ .*

**Fact 2.4** ([DEP, Proposition 5.1]). *A square free Hodge algebra over a reduced ring is reduced. In particular, an ASL over a field is reduced.*

The next result easily follows from the definition of an ASL.

**Fact 2.5.** *If  $A$  is an ASL generated by a poset  $P$  and  $x$  is the unique minimal element of  $P$ , then  $x$  is a non-zero-divisor (NZD for short) of  $A$ .*

**Fact 2.6** (see e.g. [BV, (5.2) Proposition]). *If  $A$  is an ASL generated by a poset  $P$  and  $Q_1, \dots, Q_n$  are poset ideals of  $P$ , then*

$$(Q_1 \cap \dots \cap Q_n)A = Q_1A \cap \dots \cap Q_nA.$$

**Fact 2.7** ([DEP, Proposition 1.1]). *If  $A$  is a graded Hodge algebra, then the straightening relations give a presentation of  $A$ .*

Like [Sta, II.5], we make the following

**Definition 2.8.** For a Hodge algebra  $A$  over  $k$  generated by  $H$  governed by  $\Sigma$ , we define

$$\begin{aligned} \text{core } H &:= \bigcup_{\substack{N \text{ is a generator of } \Sigma}} \text{supp } N, \\ \text{core } \Sigma &:= \{\mu \in \Sigma \mid \text{supp } \mu \subseteq \text{core } H\}, \\ \text{core } A &:= A/(H \setminus \text{core } H)A. \end{aligned}$$

It is obvious that if  $\Omega = \{x_1, \dots, x_t\}$  is a subset of  $H$  such that  $\Omega \cap \text{core } H = \emptyset$ , then  $\Omega$  is a standard subset of  $H$  and  $x_1, \dots, x_t$  is an  $A$ -regular sequence. In particular,

**Lemma 2.9.** *core  $A$  is a Hodge algebra generated by core  $H$  governed by core  $\Sigma$ . Furthermore, if  $H \setminus \text{core } H = \{x_1, \dots, x_t\}$ , then  $x_1, \dots, x_t$  is an  $A$ -regular sequence and  $\text{core } A = A/(x_1, \dots, x_t)$ .*

Moreover, it is easily verified that

$$(\text{core } A)_{\text{dis}} = \text{core}(A_{\text{dis}}).$$

So we denote both sides by  $\text{core } A_{\text{dis}}$ .

### 3 Examples of graded Hodge algebras whose discrete counterparts have strictly smaller depth

In this section, we give a series of examples of Hodge algebras whose discrete counterparts have strictly smaller depths than the original one.

Let  $n$  be a positive integer and  $X = (X_{ij})$  an  $n \times n$  generic symmetric matrix, i.e.,  $X_{ij}$  with  $1 \leq i \leq j \leq n$  are independent indeterminates and  $X_{ji} = X_{ij}$  if  $j > i$ .

DeConcini and Procesi [DP, Section 5] essentially constructed a structure of graded Hodge algebra on  $k[X] = k[X_{ij} \mid 1 \leq i \leq j \leq n]$  over  $k$ , where  $k$  is a commutative ring by the following way.

Set

$$H := \{[a_1, \dots, a_t] \mid a_i \in \mathbf{N} \text{ and } 1 \leq a_1 < \dots < a_t \leq n\}.$$

For an element  $\alpha = [a_1, \dots, a_t] \in H$ , we define the size of  $\alpha$  to be  $t$ . And for  $\alpha = [a_1, \dots, a_t]$  and  $\beta = [b_1, \dots, b_s] \in H$ , we define the relation  $\leq$  on  $H$  by

$$\alpha \leq \beta$$

if and only if

$$t \geq s \text{ and } a_i \leq b_i \text{ for } i = 1, \dots, s.$$

It is easily verified that this relation defines a partial order on  $H$ . Now set

$$D_0 := \{[\alpha|\beta] \mid \alpha, \beta \in H \text{ and } \alpha \text{ and } \beta \text{ have the same size.}\}$$

and define the relation  $<$  on  $D_0$  by

$$[\alpha|\beta] < [\gamma|\delta]$$

if and only if one of the following conditions is satisfied.

- $\alpha < \gamma$  in  $H$ .
- $\alpha = \gamma$  and  $\beta < \delta$  in  $H$ .

Then it is easily verified that this relation defines a partial order on  $D_0$ .

Let  $\Sigma_0$  be the ideal of monomials on  $D_0$  generated by

$$\{[\alpha|\beta] \mid \alpha \not\leq \beta\} \cup \{[\alpha|\beta][\gamma|\delta] \mid \beta \not\leq \gamma \text{ and } \delta \not\leq \alpha\}.$$

Then

**Theorem 3.1 ([DP, Section 5]).**  *$k[X]$  is a graded Hodge algebra over  $k$  generated by  $D_0$  governed by  $\Sigma_0$  with structure map*

$$D_0 \ni [a_1, \dots, a_t | b_1, \dots, b_t] \longmapsto \det(X_{a_i b_j}) \in k[X].$$

Since one can delete an element which is a generator of the ideal of monomials governing a Hodge algebra from the poset generating it, if we set

$$D := \{[\alpha|\beta] \in D_0 \mid \alpha \leq \beta\}$$

and

$$\Sigma := \{\mu \in \Sigma_0 \mid \text{supp } \mu \subseteq D\},$$

we see

**Corollary 3.2.**  *$k[X]$  is a graded Hodge algebra over  $k$  generated by  $D$  governed by  $\Sigma$ .*

Set

$$\Omega_t := \{[\alpha|\beta] \in D \mid \text{size } \alpha \geq t\} \quad \text{and} \quad \Omega'_t := \{[\alpha|\beta] \in D_0 \mid \text{size } \alpha \geq t\}.$$

Then  $\Omega_t$  ( $\Omega'_t$  resp.) is a poset ideal of  $D$  ( $D_0$  resp.) and

$$\Omega_t k[X] = \Omega'_t k[X] = I_t(X).$$

So by Fact 2.3, we see the following

**Corollary 3.3.**  *$k[X]/I_t(X)$  is a Hodge algebra over  $k$  generated by  $D \setminus \Omega_t$  governed by  $\Sigma/\Omega_t$ .*

From now on, we assume that  $k$  is a field. Then by the result of Kutz [Kut] (see also [Con]),  $k[X]/I_t(X)$  is Cohen-Macaulay for any  $t$ .

Let us focus our attention to the case where  $t = 2$  and  $n \geq 3$ . Set  $A := k[X]/I_2(X)$ . Since

$$D \setminus \Omega_2 = \{[i|j] \mid 1 \leq i \leq j \leq n\}$$

and  $\Sigma/\Omega_2$  is generated by  $\{[i|j][k|l] \mid j > k \text{ and } l > i\}$ , the discrete counterpart  $A_{\text{dis}}$  of  $A$  is isomorphic to

$$k[X]/I$$

where  $I$  is the monomial ideal generated by  $\{X_{ij}X_{kl} \mid j > k \text{ and } l > i\}$ . It is easily verified that  $\sqrt{I} = (X_{ij} \mid 1 \leq i < j \leq n)$ , so in particular  $A_{\text{dis}}/\sqrt{(0)}$  is the polynomial ring in  $n$  variables over  $k$ . Therefore we see that

$$\dim A = \dim A_{\text{dis}} = n.$$

In order to study the depth of  $A_{\text{dis}}$ , we use the technique of polarization (see [SV, p. 107]). Let  $Y_{ij}$  ( $1 \leq i < j \leq n$ ) be  $\binom{n}{2}$  new variables and

$$k[X, Y] = k[X_{ij}, Y_{kl} \mid 1 \leq i < j \leq n, 1 \leq k < l \leq n]$$

be the polynomial ring. Then the polarization of  $A_{\text{dis}}$  is

$$k[X, Y]/J,$$

where  $J$  is the monomial ideal generated by

$$\{X_{ij}Y_{ij} \mid 1 \leq i < j \leq n\} \cup \{X_{ij}X_{kl} \mid (i, j) \neq (k, l), j > k, l > i\}.$$

It is known that  $X_{12} - Y_{12}, X_{13} - Y_{13}, \dots, X_{1n} - Y_{1n}, X_{23} - Y_{23}, \dots, X_{n-1,n} - Y_{n-1,n}$  is a  $k[X, Y]/J$ -regular sequence (see [DEP, Proposition 4.3]) and

$$k[X, Y]/(J + (X_{ij} - Y_{ij} \mid 1 \leq i < j \leq n)) = A_{\text{dis}}.$$

So

$$\dim A_{\text{dis}} - \text{depth } A_{\text{dis}} = \dim(k[X, Y]/J) - \text{depth}(k[X, Y]/J).$$

Since  $J$  is a square-free monomial ideal, it corresponds to a simplicial complex by the theory of Stanley-Reisner rings. It is easily verified that

$$\{X_{11}, X_{1n}, X_{nn}\} \cup \{Y_{ij} \mid 1 \leq i < j \leq n, (i, j) \neq (1, n)\}$$

and

$$\{X_{ij} \mid 1 \leq i \leq j \leq n, j \leq i + 1\} \cup \{Y_{ij} \mid 1 \leq i, i + 2 \leq j \leq n\}$$

are facets of this simplicial complex. So it has facets of dimensions  $\binom{n+1}{2} - 1$  and  $\binom{n}{2} + 1$ . Since

$$\left(\binom{n+1}{2} - 1\right) - \left(\binom{n}{2} + 1\right) = n - 2,$$



we see, by the theory of Stanley-Reisner rings, that

$$\dim(k[X, Y]/J) - \text{depth}(k[X, Y]/J) \geq n - 2$$

(see e.g. [Miy2, p. 370]). Therefore

$$\text{depth } A_{\text{dis}} \leq 2$$

since  $\dim A_{\text{dis}} = n$ .

On the other hand, since no generator of  $I$  involve  $X_{11}$  and  $X_{nn}$ , we see that  $X_{11}, X_{nn}$  is an  $A_{\text{dis}}$ -regular sequence. So

$$\text{depth } A_{\text{dis}} = 2.$$

Summing up,  $A$  is a Cohen-Macaulay homogeneous Hodge algebra of dimension  $n$  and the discrete counterpart  $A_{\text{dis}}$  has depth 2. Moreover,  $A$  is generated by  $D \setminus \Omega_2$  and  $\text{core}(D \setminus \Omega_2) = (D \setminus \Omega_2) \setminus \{[1|1], [n|n]\}$ . Therefore,  $\text{core } A$  is a Cohen-Macaulay homogeneous Hodge algebra of dimension  $n - 2$  and  $\text{depth}(\text{core } A_{\text{dis}}) = 0$ .

**Remark 3.4.** It is known that  $A$  is isomorphic to the second Veronese subring of the polynomial ring in  $n$  variables over  $k$ . So by [Mok],  $A$  is Gorenstein when  $n$  is an even number.

**Remark 3.5.** As is noted above, we can give a Hodge algebra structure on the polynomial ring  $k[X]$ . It is also verified by the same way that the discrete counterpart of this Hodge algebra has strictly smaller depth than that of  $k[X]$  if  $n \geq 3$ .

## 4 Stepping stones

In the following of this article, we focus our attention to ASL and consider the following

**Problem 4.1.** *If there is a Cohen-Macaulay ASL over  $k$  generated by a poset  $P$ , what can be said about  $P$ ? In particular, is  $P$  Cohen-Macaulay over  $k$ ?*

To tackle this problem, we state two stepping stones and consider the following four conditions.

- (i)  $P$  is a poset.

- (ii)  $P$  is a pure poset.
- (iii)  $P$  is a Buchsbaum poset.
- (iv)  $P$  is a Cohen-Macaulay poset.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are well known. And in the following, we state that, under the assumption that there is a Cohen-Macaulay ASL generated by  $P$ , (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii) are also valid.

Let  $k$  be a field,  $X_1, \dots, X_n$  be indeterminates over  $k$  and  $S = k[X_1, \dots, X_n]$  a polynomial ring. Assume that  $S$  is given a graded ring structure such that  $S_0 = k$  and each  $X_i$  is a homogeneous element of positive degree.

For a graded ideal  $I$  of  $S$ , Hartshorne [Har], Sturmfels-Trung-Vogel [STV] defined the notion of geometric degree  $\text{geom-deg } I$  and arithmetic degree  $\text{arith-deg } I$  of  $I$ . By definition

$$\deg I \leq \text{geom-deg } I \leq \text{arith-deg } I$$

and

$$\deg I = \text{geom-deg } I \iff S/I \text{ is equidimensional,}$$

$$\text{geom-deg } I = \text{arith-deg } I \iff S/I \text{ has no embedded prime ideals.}$$

Assume a monomial order on  $S$  is settled and let  $\text{in}(I)$  be the initial ideal of  $I$  with respect to this monomial order. It is well known that  $\deg I = \deg(\text{in}(I))$ . And Hartshorne [Har], Sturmfels-Trung-Vogel [STV] showed that

$$\text{geom-deg}(\text{in}(I)) \leq \text{geom-deg } I$$

and

$$\text{arith-deg}(\text{in}(I)) \geq \text{arith-deg } I.$$

So if  $S/\text{in}(I)$  has no embedded prime ideals, then

$$\text{geom-deg}(\text{in}(I)) = \text{geom-deg } I = \text{arith-deg } I = \text{arith-deg}(\text{in}(I))$$

and  $S/I$  is equidimensional if and only if  $S/\text{in}(I)$  is equidimensional.

Assume that  $A$  is a graded Hodge algebra over  $k$ . Then it is well known that there is a polynomial ring  $S$  with monomial order and a graded ideal  $I$  of  $S$  such that

$$A \simeq S/I \quad \text{and} \quad A_{\text{dis}} \simeq S/\text{in}(I).$$

Assume further that  $A$  is square-free. Then it is known [DEP, Proposition 5.1] that  $A$  and  $A_{\text{dis}}$  are reduced rings. So by the arguments above, we see the following

**Proposition 4.2.** *Let  $A$  be a square-free graded Hodge algebra over a field. Then  $A$  is equidimensional if and only if  $A_{\text{dis}}$  is equidimensional. In particular, if  $A$  is a graded ASL generated by a poset  $P$ , then  $A$  is equidimensional if and only if  $P$  is pure.*

Since a Cohen-Macaulay ring is equidimensional, we see the following

**Corollary 4.3.** *Let  $P$  be a poset. If there is a Cohen-Macaulay ASL generated by  $P$ , then  $P$  is pure.*

## 5 A Buchsbaum poset which generate a Cohen-Macaulay ASL is Cohen-Macaulay

In this section, we shall prove that if a poset  $P$  generates a Cohen-Macaulay ASL and if  $P$  itself is Buchsbaum, then  $P$  is Cohen-Macaulay.

We begin by noting the graded version of the Theorem of Huckaba and Marley [HM], which is proved by noting Lemma 5.2 below and reducing the proof of [HM, Proposition 3.2] to the case of bigraded prime ideals.

**Theorem 5.1 (The graded version of the Theorem of Huckaba-Marley).**

*Let  $A$  be a non-negatively graded Noetherian ring with  $A_0$  a field and  $I$  a non-nilpotent graded ideal of  $A$ . Denote by  $R$  the Rees algebra with respect to  $I$  and by  $G$  the associated graded ring. Suppose that*

$$\text{depth } G < \text{depth } A.$$

*Then*

$$\text{depth } R = \text{depth } G + 1.$$

**Lemma 5.2.** *Let  $R$  be an  $\mathbf{N}^2$ -graded ring and  $I$  an ideal of  $R$  which is homogeneous in the first grading. If we set*

$$I^* := (a \in I \mid a \text{ is homogeneous in the second grading}),$$

*then*

$$I^* = (x \in I \mid x \text{ is bihomogeneous}).$$

*In particular,  $I^*$  is a bigraded ideal.*

Next we state the following lemma due to Takeshi Kawasaki and Kazuhiko Kurano.

**Lemma 5.3.** *Let  $B$  be an  $\mathbf{N}^m$ -graded Noetherian ring,  $M$  a finitely generated graded  $B$ -module. Then for any  $i \in \mathbf{Z}$  and  $\alpha \in \mathbf{Z}^m$ ,*

$$[H_N^i(M)]_\alpha$$

*is a finitely generated  $B_{(0,\dots,0)}$ -module, where  $N = \bigoplus_{\alpha \in \mathbf{N}^m \setminus \{(0,\dots,0)\}} B_\alpha$ .*

**Proof.** We may assume that  $B = B_{(0,\dots,0)}[X_1, \dots, X_l]$ , a polynomial ring with  $\deg X_i \in \mathbf{N}^m \setminus \{(0, \dots, 0)\}$ . Then  $N = (X_1, \dots, X_l)$  and we can compute the local cohomology with support  $N$  by the Čech complex with respect  $X_1, \dots, X_l$ .

Since

$$H_N^i(B) = \begin{cases} 0 & \text{if } i \neq l \\ X_1^{-1} \dots X_l^{-1} B_{(0,\dots,0)}[X_1^{-1}, \dots, X_l^{-1}] & \text{if } i = l \end{cases}$$

the assertion holds if  $M$  is a free module.

Now we prove the assertion by the backward induction on  $i$ .

If  $i > l$ , then  $H_N^i(M) = 0$  since  $H_N^i(M)$  can be computed by the Čech complex with respect to  $X_1, \dots, X_l$ .

Now assume that  $i \leq l$ . Take a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0 \quad (5.1)$$

in the category of graded  $B$ -modules with  $F$  a free  $B$ -module of finite rank. Then  $[H_N^i(F)]_\alpha$  and  $[H_N^{i+1}(K)]_\alpha$  are finitely generated  $B_{(0,\dots,0)}$ -module by the fact that  $F$  is free and the inductive hypothesis. So from the long exact sequence obtained by (5.1), we see that  $[H_N^i(M)]_\alpha$  is a finitely generated  $B_{(0,\dots,0)}$ -module. ■

Next we state the following

**Lemma 5.4.** *Let  $k$  be an infinite field and  $G$  an  $\mathbf{N}^m$ -graded Hodge algebra over  $k$ . Then for any  $\alpha \in \mathbf{Z}^m$ ,  $[H_N^i(G)]_\alpha$  is a subquotient of  $[H_{N'}^i(G_{\text{dis}})]_\alpha$ , where  $N$  (or  $N'$  resp.) is the unique  $\mathbf{N}^m$ -graded maximal ideal of  $G$  ( $G_{\text{dis}}$  resp.).*

**Proof.** Let  $P = \{x_1, \dots, x_n\}$  be the poset which generate the Hodge algebra  $G$  and  $f: k[X_1, \dots, X_n] \rightarrow G$  be the  $k$ -algebra homomorphism sending  $X_i$  to  $x_i$ , where  $k[X_1, \dots, X_n]$  is the polynomial ring. Set  $\ker f = I$  and  $\underline{X} = X_1, \dots, X_n$ . Then

$$k[\underline{X}]/I \simeq G.$$

It is well known that there is a monomial order on  $k[\underline{X}]$  such that

$$k[\underline{X}]/\text{in}(I) \simeq G_{\text{dis}}.$$

By the theory of Gröbner basis (see e.g. [Eis, Section 15.8]), there is an ideal  $J$  of  $k[\underline{X}, T]$  such that

- $k[\underline{X}, T]/J$  is a free  $k[T]$ -module,
- $k[\underline{X}, T]/(J, T) \simeq k[\underline{X}]/\text{in}(I)$  and
- $k[\underline{X}, T]/(J, T - u) \simeq k[\underline{X}]/I$  for any  $u \in k$  with  $u \neq 0$ ,

where  $T$  is a new variable. Put  $B = k[\underline{X}, T]$  and consider  $B$  as an  $\mathbf{N}^m$ -graded ring by setting  $\deg T = 0$  and  $\deg X_i = \deg x_i$  for  $i = 1, \dots, n$ .

Then by Lemma 5.3

$$[H_{(\underline{X})}^i(k[\underline{X}, T]/J)]_\alpha$$

is a finitely generated  $k[T]$ -module for any  $i \in \mathbf{Z}$  and  $\alpha \in \mathbf{Z}^m$ . So we may write

$$[H_{(\underline{X})}^i(k[\underline{X}, T]/J)]_\alpha = k[T]^{n_i} \oplus k[T]/(e_1) \oplus \dots \oplus k[T]/(e_r)$$

$$[H_{(\underline{X})}^{i+1}(k[\underline{X}, T]/J)]_\alpha = k[T]^{n_{i+1}} \oplus k[T]/(f_1) \oplus \dots \oplus k[T]/(f_s)$$

since  $k[T]$  is a principal ideal domain.

Now let  $u$  be an arbitrary element of  $k$ . By the short exact sequence

$$0 \longrightarrow k[\underline{X}, T]/J \xrightarrow{T-u} k[\underline{X}, T]/J \longrightarrow k[\underline{X}, T]/(J, T - u) \longrightarrow 0$$

we have the following long exact sequence

$$\begin{aligned} & [H_{(\underline{X})}^i(k[\underline{X}, T]/J)]_\alpha \xrightarrow{T-u} [H_{(\underline{X})}^i(k[\underline{X}, T]/J)]_\alpha \longrightarrow [H_{(\underline{X})}^i(k[\underline{X}, T]/(J, T - u))]_\alpha \\ & \longrightarrow [H_{(\underline{X})}^{i+1}(k[\underline{X}, T]/J)]_\alpha \xrightarrow{T-u} [H_{(\underline{X})}^{i+1}(k[\underline{X}, T]/J)]_\alpha. \end{aligned}$$

So  $[H_{(\underline{X})}^i(k[\underline{X}, T]/(J, T - u))]_\alpha$  is isomorphic to  $(k[T]/(T - u))^{n_i}$ , except for finitely many  $u \in k$ . And is a subquotient of the corresponding module of any other  $u \in k$ .

Since

$$k[\underline{X}, T]/(J, T - u) \simeq G$$

and so

$$[H_N^i(G)]_\alpha \simeq [H_{(\underline{X})}^i(k[\underline{X}, T]/(J, T - u))]_\alpha$$

for any  $u \in k$  with  $u \neq 0$ ,  $[H_N^i(G)]_\alpha$  is a subquotient of  $[H_{(\underline{X})}^i(k[\underline{X}, T]/(J, T))]_\alpha \simeq [H_N^i(G_{\text{dis}})]_\alpha$ . ■

Now we state

**Theorem 5.5.** *Let  $A$  be a graded Cohen-Macaulay square-free Hodge algebra over a field  $k$ . Suppose that  $\text{core } A_{\text{dis}}$  is Buchsbaum. Then  $A_{\text{dis}}$  is Cohen-Macaulay.*

**Proof.** Let  $H$  be the poset which generate the Hodge algebra  $A$  and  $\Sigma$  the ideal of monomials on  $H$  which govern  $A$ . Set  $\Delta := \{\sigma \subseteq H \mid \prod_{x_i \in \sigma} x_i \notin \Sigma\}$ . Then  $A_{\text{dis}} = k[\Delta]$ .

In order to prove the theorem, we may assume, by tensoring an infinite field containing  $k$ , that  $k$  is an infinite field. And by considering  $\text{core } A$  instead of  $A$ , we may assume that  $A_{\text{dis}} = k[\Delta]$  is Buchsbaum.

We prove the theorem by induction on  $|\text{ind } A|$ , where  $\text{ind } A$  stands for the indiscrete part of  $A$  (cf. [DEP, p. 16]). If  $\text{ind } A = \emptyset$ , then  $A_{\text{dis}} = A$  and the assertion is clear. So we assume that  $\text{ind } A \neq \emptyset$ .

Take a minimal element  $x$  of  $\text{ind } A$  and set  $I = xA$ . Denote by  $R$  the Rees algebra with respect to  $I$  and by  $G$  the associated graded ring. Then  $R$  is a bigraded ring and  $G$  is a bigraded Hodge algebra over  $k$  such that  $\text{ind } G \subseteq \text{ind } A \setminus \{x\}$  ([DEP, Theorem 3.1]).

If  $G$  is Cohen-Macaulay, then by the inductive hypothesis, we see that  $A_{\text{dis}} = G_{\text{dis}}$  is Cohen-Macaulay. So we assume that  $G$  is not Cohen-Macaulay. Set  $\text{depth } G = e$  and  $\dim A = d$ . And let  $M$  (resp.  $\mathfrak{m}$ ) be the unique bigraded (resp. graded) maximal ideal of  $R$  (reap.  $A$ ).

Since  $R$  and  $G$  are bigraded rings, there are two entries in the degrees of these rings. But from now on, we use the notation and terminology concerning grading to mean the grading newly defined by the Rees algebra. Then, since  $A$  is concentrated in degree 0 and  $H_M^i(A) = H_{\mathfrak{m}}^i(A)$ , we see by the long exact sequence of the local cohomology modules obtained by the short exact sequence

$$0 \longrightarrow R_+ \longrightarrow R \longrightarrow A \longrightarrow 0 \quad (5.2)$$

that

$$[H_M^i(R_+)]_n \simeq [H_M^i(R)]_n \quad (5.3)$$

for any  $i, n \in \mathbf{Z}$  with  $n \neq 0$ .

On the other hand, since  $IR = R_+(1)$ , by the long exact sequence obtained by

$$0 \longrightarrow IR \longrightarrow R \longrightarrow G \longrightarrow 0$$

we see that there is an exact sequence

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & [H_M^{i-1}(G)]_n & & \\ \longrightarrow & [H_M^i(R_+)]_{n+1} & \longrightarrow & [H_M^i(R)]_n & \longrightarrow & [H_M^i(G)]_n & \\ \longrightarrow & [H_M^{i+1}(R_+)]_{n+1} & \longrightarrow & \cdots & & & \end{array} \quad (5.4)$$

for any  $i, n \in \mathbf{Z}$ .

Now we recall the following result of Hochster.

**Theorem 5.6** (see [Sta, Chapter II 4.1 Theorem]). *Let  $\Delta$  be a simplicial complex with vertex set  $\{x_1, \dots, x_n\}$ . Then the  $\mathbf{N}^n$ -graded Hilbert series of  $H_{\mathfrak{m}}^i(k[\Delta])$  is*

$$\sum_{\sigma \in \Delta} (\dim_k \tilde{H}^{i-|\sigma|-1}(\text{link}_{\Delta}(\sigma); k)) \prod_{x_i \in \sigma} \frac{\lambda_i^{-1}}{1 - \lambda_i^{-1}}$$

where  $\mathfrak{m}$  is the unique graded maximal ideal.

We return to the proof of Theorem 5.5. By Theorem 5.6 and Lemma 5.4, we see that

$$[H_M^i(G)]_n = 0 \tag{5.5}$$

if  $n > 0$ . So we see by (5.4) that the map

$$[H_M^i(R_+)]_{n+1} \longrightarrow [H_M^i(R)]_n$$

is an epimorphism for any  $i, n \in \mathbf{Z}$  with  $n > 0$ . On the other hand,

$$[H_M^i(R)]_n = 0 \quad \text{for } n \gg 0,$$

since  $H_M^i(R)$  is an Artinian module. Therefore, we see that

$$[H_M^i(R_+)]_n \simeq [H_M^i(R)]_n = 0$$

for any  $i, n \in \mathbf{Z}$  with  $n > 0$ .

So by (5.4), we see that

$$[H_M^e(G)]_0 \simeq [H_M^e(R)]_0.$$

Since  $\text{depth } R = \text{depth } G + 1 = e + 1$  by Theorem 5.1, it follows that

$$[H_M^e(G)]_0 \simeq [H_M^e(R)]_0 = 0.$$

On the other hand,  $e = \text{depth } G$  by definition, so by (5.5) we see that there is an  $n \in \mathbf{Z}$  such that

$$[H_M^e(G)]_n \neq 0 \quad \text{and} \quad n < 0.$$

It follows from Theorem 5.6 and Lemma 5.4 that

$$\tilde{H}^{e-2}(\text{link}_{\Delta}(x); k) \neq 0.$$

But this contradicts to the assumption that  $A_{\text{dis}} = k[\Delta]$  is Buchsbaum. (For characterizations of Buchsbaum complexes, see e.g. [Miy1].) ■

**Remark 5.7.** Let  $A$  be the Hodge algebra, of the case  $n = 3$ , considered in section 3. Then, as is easily verified,

$$\text{core } A_{\text{dis}} = k[X_{12}, X_{13}, X_{22}, X_{23}]/I$$

where

$$I = (X_{12}^2, X_{12}X_{13}, X_{13}^2, X_{13}X_{22}, X_{13}X_{23}, X_{23}^2).$$

Since

$$I = (X_{12}, X_{13}, X_{22}, X_{23})^2 \cap (X_{12}^2, X_{13}, X_{23}^2)$$

and

$$I : (X_{12}, X_{13}, X_{22}, X_{23}) = (X_{12}^2, X_{13}, X_{23}^2)$$

is the primary component corresponding to the unique minimal prime ideal  $(X_{12}, X_{13}, X_{23})$  of  $I$ , it is easily verified that

$$I : a = (X_{12}^2, X_{13}, X_{23}^2)$$

for any system of parameter  $a$  of  $\text{core } A_{\text{dis}}$ . So  $\text{core } A_{\text{dis}}$  is Buchsbaum. And as is noted in section 3,  $A$  is Cohen-Macaulay. Therefore the square-free hypothesis in Theorem 5.5 is essential.

## 6 Rees algebras

As is noted below, a Rees algebra of an ASL have a structure of an ASL under certain conditions. In this section we study the relation between the Cohen-Macaulay property of such a Rees algebra and the property of the discrete counterpart of it.

First we recall the definition of a straightening closed ideal.

**Definition 6.1.** Let  $A$  be a graded ASL over a field  $k$  generated by a poset  $P$  and  $Q$  a poset ideal of  $P$ . If every standard monomial  $\mu_i$  appearing in the standard representation

$$\alpha\beta = \sum_i r_i \mu_i, \quad 0 \neq r_i \in k$$

of  $\alpha\beta$  with  $\alpha, \beta \in Q$  with  $\alpha \not\prec \beta$ , has at least two factors in  $Q$ , we say that  $Q$  (or the ideal  $QA$  of  $A$ ) is straightening closed.

Note that any poset ideal of a discrete ASL is straightening closed.

Now let  $P$  be a poset and  $Q$  a poset ideal of  $P$ . We define the poset  $P \uplus Q$  as follows (cf. [BV, Section 9]). Denote a copy of  $Q$  by  $Q^*$  and the element corresponding to  $x \in Q$  by  $x^* \in Q^*$ . Set  $P \uplus Q = P \cup Q^*$  as the underlying set. And for  $\alpha, \beta \in P \uplus Q$ , we define  $\alpha < \beta$  if and only if one of the following three conditions is satisfied.



- $\alpha, \beta \in P$  and  $\alpha < \beta$  in  $P$ .
- $\alpha = x^*, \beta = y^*$  with  $x, y \in Q$  and  $x < y$  in  $P$ .
- $\alpha = x^*$  with  $x \in Q, \beta \in P$  and  $x \leq \beta$  in  $P$ .

With this notation, we recall the following fact (see [DEP, 10d] or [BV, (9.13)]).

**Proposition 6.2.** *Let  $A$  be a graded ASL over a field  $k$  generated by a poset  $P$ . Suppose that  $Q$  is a straightening closed poset ideal of  $P$  and  $I = QA$ . Then*

- The Rees algebra  $R$  with respect to  $I$  is a graded ASL over  $k$  generated by  $P \uplus Q$ .*
- The associated graded ring  $G$  is a graded ASL over  $k$  generated by  $P$  such that  $\text{ind } G \subseteq \text{ind } A$ . In particular, if  $A$  is the discrete ASL, then so is  $G$ .*

Now we examine the Cohen-Macaulay property of  $P \uplus Q$ , where  $P$  is a poset and  $Q$  is a poset ideal of  $P$ .

**Theorem 6.3.** *Let  $P$  be a Cohen-Macaulay poset over a field  $k$  and  $Q$  a poset ideal of  $P$ . If*

$$\tilde{\chi}((-\infty, x)_P) = 0 \quad \text{for any } x \in (P \cup \{\infty\}) \setminus Q \quad (6.1)$$

*then  $P \uplus Q$  is also Cohen-Macaulay over  $k$ .*

In order to prove the theorem, we state several lemmas first.

**Lemma 6.4.** *A poset  $P$  is Cohen-Macaulay over a field  $k$  if and only if*

$$\tilde{H}^i(\Delta((x, y)_P); k) = 0$$

*for any  $x, y \in P \cup \{-\infty, \infty\}$  with  $x < y$  and any  $i \in \mathbf{Z}$  with  $i < \dim \Delta((x, y)_P)$ .*

This Lemma follows from Reisner's criterion of Cohen-Macaulay property [Rei] and induction.

**Lemma 6.5.** *Suppose  $P$  is a poset and  $Q$  is a poset ideal of  $P$ . Then the natural inclusion*

$$\Delta(P) \subseteq \Delta(P \uplus Q)$$

*induces a deformation retract of the geometric realizations. In particular, the induced maps*

$$\tilde{H}^i(\Delta(P \uplus Q); k) \longrightarrow \tilde{H}^i(\Delta(P); k)$$

*between the cohomologies are isomorphisms.*

**Proof.** Let  $X$  be a geometric realization of  $\Delta(P)$ . Then a geometric realization of  $\Delta(P \uplus Q)$  can be realized as a subset of  $X \times I$ , where  $I$  is the closed interval  $[0, 1]$ , by the following way.

Correspond  $\alpha \in P \uplus Q$  with  $\alpha = x^*$ ,  $x \in Q$  the point  $(x^*, 1) \in X \times I$ , where  $x^*$  denote the image of  $x$  in  $X$ . And correspond  $\alpha \in P \uplus Q$  with  $\alpha \in P$  the point  $(\alpha^*, 0) \in X \times I$ . It is easily verified that it really defines a geometric realization. Denote the geometric realization of  $\Delta(P \uplus Q)$  defined above by  $Y$ . Then it is also easily verified that  $(\xi, a) \in Y$  and  $0 \leq b \leq a$  imply  $(\xi, b) \in Y$ , and so

$$\begin{aligned} r: Y &\longrightarrow X \\ (\xi, a) &\longmapsto \xi \end{aligned}$$

is a retraction. ■

**Lemma 6.6.** *Let  $P$  be a poset with unique minimal element  $x_0$  and  $Q$  a non-empty poset ideal of  $P$ . Then*

$$\tilde{H}^i(\Delta((P \uplus Q) \setminus \{x_0^*\}); k) = 0$$

for any  $i \in \mathbf{Z}$ .

**Proof.** Set  $\Pi = P \uplus Q \setminus \{x_0^*\}$ ,  $\Pi_1 = \{\alpha \in \Pi \mid \alpha \geq x_0\}$ ,  $\Pi_2 = \Pi \setminus \{x_0\}$  and  $\Pi_3 = \Pi_1 \cap \Pi_2$ . Then, since  $\Pi = \Pi_1 \cup \Pi_2$ , we have the following Mayer-Vietoris sequence of cohomologies.

$$\begin{array}{ccccccc} & \cdots & \longrightarrow & \tilde{H}^{i-1}(\Delta(\Pi_1); k) \oplus \tilde{H}^{i-1}(\Delta(\Pi_2); k) & \longrightarrow & \tilde{H}^{i-1}(\Delta(\Pi_3); k) & \\ \longrightarrow & \tilde{H}^i(\Delta(\Pi); k) & \longrightarrow & \tilde{H}^i(\Delta(\Pi_1); k) \oplus \tilde{H}^i(\Delta(\Pi_2); k) & \longrightarrow & \tilde{H}^i(\Delta(\Pi_3); k) & \\ \longrightarrow & \cdots & & & & & \end{array}$$

Since  $\Pi_2 = (P \setminus \{x_0\}) \uplus (Q \setminus \{x_0\})$  and  $\Pi_3 = P \setminus \{x_0\}$ , we see, by Lemma 6.5, that the map

$$\tilde{H}^j(\Delta(\Pi_2); k) \longrightarrow \tilde{H}^j(\Delta(\Pi_3); k)$$

in the Mayer-Vietoris sequence is an isomorphism for any  $j$ . So we see that

$$\tilde{H}^i(\Delta(\Pi); k) \simeq \tilde{H}^i(\Delta(\Pi_1); k)$$

for any  $i \in \mathbf{Z}$ .

Since  $\Pi_1 = P$  and  $P$  has a unique minimal element, we see that

$$\tilde{H}^i(\Delta(\Pi); k) \simeq \tilde{H}^i(\Delta(\Pi_1); k) = 0$$

for any  $i \in \mathbf{Z}$ . ■

Now we prove Theorem 6.3.

Set  $\Pi = P \uplus Q$  and we prove that

$$\tilde{H}^i(\Delta((\alpha, \beta)_\Pi); k) = 0$$

for any  $\alpha, \beta \in \Pi \cup \{-\infty, \infty\}$  with  $\alpha < \beta$  and any  $i \in \mathbf{Z}$  with  $i < \dim \Delta((\alpha, \beta)_\Pi)$ .

The case where  $\alpha \in P$  or  $\beta = y^*$  with  $y \in Q$  are clear from the Cohen-Macaulay property of  $P$ .

The case where  $\alpha = x^*$  with  $x \in Q$  and  $\beta = y$  with  $y \in P \cup \{\infty\}$ . If we put  $P' = [x, y)_P$  and  $Q' = P' \cap Q$ , then  $P'$  is a poset with unique minimal element  $x$  and  $Q'$  is a poset ideal of  $P'$ . Furthermore

$$(\alpha, \beta)_\Pi = (P' \uplus Q') \setminus \{x^*\}.$$

So the result follows from Lemma 6.6.

The case where  $\alpha = -\infty$  and  $\beta = y$  with  $y \in Q$  is verified by considering the poset anti-isomorphic to  $Q$  and using Lemma 6.6.

The only remaining case is  $\alpha = -\infty$  and  $\beta = y$  with  $y \in (P \cup \{\infty\}) \setminus Q$ . Set  $P' = (-\infty, y)_P$  and  $Q' = P' \cap Q$ . Then  $Q'$  is a poset ideal of  $P'$  and

$$(\alpha, \beta)_\Pi = P' \uplus Q'.$$

So by Lemma 6.5, we see that

$$\tilde{H}^i(\Delta((\alpha, \beta)_\Pi); k) \simeq \tilde{H}^i(\Delta(P'); k)$$

for any  $i \in \mathbf{Z}$ . Since  $P$  is Cohen-Macaulay over  $k$ , we see that

$$\tilde{H}^i(\Delta(P'); k) = 0$$

for any  $i \in \mathbf{Z}$  with  $i < \dim \Delta(P')$ . On the other hand,

$$\tilde{\chi}(\Delta(P')) = 0$$

by assumption, so we see that

$$\tilde{H}^i(\Delta(P'); k) = 0$$

for any  $i \in \mathbf{Z}$ . Therefore

$$\tilde{H}^i(\Delta((\alpha, \beta)_\Pi); k) = 0$$

for any  $i \in \mathbf{Z}$ . ■

It follows directly from Theorem 6.3 the following

**Corollary 6.7.** *If  $P$  is a Cohen-Macaulay poset over a field  $k$  with unique minimal element and  $Q$  is a poset ideal of  $P$ , then  $P \uplus Q$  is also Cohen-Macaulay over  $k$ .*

Note the posets considered by Bruns-Vetter in [BV, Section 9] are Cohen-Macaulay posets with unique minimal element. So Corollary 6.7 gives another proof of [BV, (9.4) Theorem (b)].

We consider the relation between the above result and Cohen-Macaulay property of Rees algebras in the sequel of this section. First we recall the result of Trung-Ikeda [TI].

**Theorem 6.8.** *Let  $(A, \mathfrak{m})$  be a Cohen-Macaulay local ring and  $I$  a proper, non-nilpotent ideal of  $A$ . Denote the Rees algebra with respect to  $I$  by  $R$  and the associated graded ring by  $G$ . Then the following are equivalent.*

- (i)  $R$  is Cohen-Macaulay.
- (ii)  $G$  is Cohen-Macaulay and  $a(G) < 0$ .

(See [GW] for the definition of the  $a$ -invariant  $a(G)$ .) It is verified, by the same way as Theorem 5.1, that the same conclusion follows if one assumes that  $A$  is a non-negatively graded ring over a field and  $I$  is a graded ideal of  $A$ . But in this case, one have to be careful enough to interpret the  $a$ -invariant of  $G$  to be defined by the grading newly defined by the Rees algebra, not by the original grading of  $A$ .

Now let us examine when the  $a$ -invariant of  $G$  is negative, in case  $A$  is a graded ASL over a field  $k$  generated by a Cohen-Macaulay poset  $P$ ,  $Q$  is a straightening closed poset ideal of  $P$  and  $I = QA$ .

By setting the original grading of  $A$  as the first coordinate and the grading of Rees algebra as the second coordinate, we have an  $\mathbf{N}^2$ -grading of  $G$ . It is well known that the bigraded Hilbert series

$$H_G(\lambda, \mu) := \sum_{i,j} (\dim_k G_{ij}) \lambda^i \mu^j$$

is a rational function, and if

$$H_G(\lambda, \mu) = \frac{g(\lambda, \mu)}{f(\lambda, \mu)},$$

where  $f$  and  $g$  are polynomials, then, since  $G$  is Cohen-Macaulay,

$$a(G) < 0 \iff \deg_\mu g < \deg_\mu f,$$

where  $\deg_\mu$  stands for the degree with respect to  $\mu$ . On the other hand, an ASL and the discrete counterpart have the same Hilbert function. Therefore

$$a(G) < 0 \iff a(G_{\text{dis}}) < 0.$$

Set  $P = \{x_1, \dots, x_n\}$  and  $Q = \{x_1, \dots, x_m\}$ . Then by setting

$$\deg x_i = \begin{cases} (0, \dots, 0, 1, 0, \dots, 0, 1) & \text{if } i \leq m \text{ (1 in the } i\text{-th and the last position)} \\ (0, \dots, 0, 1, 0, \dots, 0, 0) & \text{if } i > m \text{ (1 in the } i\text{-th position)} \end{cases}$$

we can make  $G_{\text{dis}}$  an  $\mathbf{N}^{n+1}$ -graded ring. The  $\mathbf{N}^{n+1}$ -graded Hilbert series  $H_{G_{\text{dis}}}(\lambda_1, \dots, \lambda_n, \mu)$  of  $G_{\text{dis}}$  is

$$\frac{\sum_{\sigma \in \Delta(P)} \left( \prod_{i \leq m}^{x_i \in \sigma} \lambda_i \mu \right) \left( \prod_{i \leq m}^{x_i \notin \sigma} (1 - \lambda_i \mu) \right) \left( \prod_{i > m}^{x_i \in \sigma} \lambda_i \right) \left( \prod_{i > m}^{x_i \notin \sigma} (1 - \lambda_i) \right)}{\prod_{i \leq m} (1 - \lambda_i \mu) \prod_{i > m} (1 - \lambda_i)}$$

(see e.g. [Sta, II 1.4 Theorem]).

The  $\mu$ -degree of the denominator is  $m$  and the  $\mu$ -degree of the numerator is at most  $m$ . So  $a(G_{\text{dis}}) < 0$  if and only if the coefficient of  $\mu^m$  of the numerator is zero.

The coefficient of  $\mu^m$  of the numerator is

$$\begin{aligned} & \prod_{i=1}^m \lambda_i \sum_{\sigma \in \Delta(P)} (-1)^{|Q \setminus \sigma|} \left( \prod_{\substack{x_i \in \sigma \\ i > m}} \lambda_i \right) \left( \prod_{\substack{x_i \notin \sigma \\ i > m}} (1 - \lambda_i) \right) \\ &= (-1)^{|Q|} \prod_{i=1}^m \lambda_i \sum_{\substack{\tau \in \Delta(P) \\ \tau \cap Q = \emptyset}} \left( \sum_{\substack{v \in \Delta(Q) \\ v \cup \tau \in \Delta(P)}} (-1)^{|v|} \right) \left( \prod_{x_i \in \tau} \lambda_i \right) \left( \prod_{\substack{x_i \notin \tau \\ i > m}} (1 - \lambda_i) \right) \\ &= (-1)^{|Q|} \prod_{i=1}^m \lambda_i \sum_{\substack{\tau \in \Delta(P) \\ \tau \cap Q = \emptyset}} \left( \prod_{x_i \in \tau} \lambda_i \right) \left( \prod_{\substack{x_i \notin \tau \\ i > m}} (1 - \lambda_i) \right) \tilde{\chi}(\{y \in Q \mid y < \min(\tau \cup \{\infty\})\}) \end{aligned}$$

Since  $\left( \prod_{x_i \in \tau} \lambda_i \right) \left( \prod_{x_i \notin \tau, i > m} (1 - \lambda_i) \right)$  are independent polynomials, we see that  $a(G) < 0$  if and only if  $\tilde{\chi}(\{y \in Q \mid y < \min(\tau \cup \{\infty\})\}) = 0$  for any  $\tau \in \Delta(P)$  with  $\tau \cap Q = \emptyset$ . It is easily verified that the last condition is equivalent to

$$\tilde{\chi}(\{y \in Q \mid y < x\}) = 0 \quad \text{for any } x \in (P \cup \{\infty\}) \setminus Q.$$

So we see the following

**Lemma 6.9.** *In the setting above,*

$$a(G) < 0$$

*if and only if*

$$\tilde{\chi}(\{y \in Q \mid y < x\}) = 0 \quad \text{for any } x \in (P \cup \{\infty\}) \setminus Q.$$

Next we state the following

**Lemma 6.10.** *Let  $P$  be a poset and  $Q$  a poset ideal of  $P$ . Then the following conditions are equivalent.*

- (i)  $\tilde{\chi}(\{y \in Q \mid y < x\}) = 0$  for any  $x \in (P \cup \{\infty\}) \setminus Q$ .
- (ii)  $\tilde{\chi}((-\infty, x)_P) = 0$  for any  $x \in (P \cup \{\infty\}) \setminus Q$ .

**Proof.** We first note that

$$\begin{aligned} & \tilde{\chi}((-\infty, x)_P) \\ &= \sum_{\sigma \in \Delta((-\infty, x)_P)} (-1)^{|\sigma|-1} \\ &= \sum_{\sigma \in \Delta((-\infty, x)_P), \sigma \subseteq Q} (-1)^{|\sigma|-1} + \sum_{\sigma \in \Delta((-\infty, x)_P), \sigma \not\subseteq Q} (-1)^{|\sigma|-1} \\ &= \tilde{\chi}(\{y \in Q \mid y < x\}) + \sum_{\emptyset \neq \tau \in \Delta((-\infty, x)_P \setminus Q)} (-1)^{|\tau|} \sum_{v \in Q, \tau \cup v \in \Delta((-\infty, x)_P)} (-1)^{|v|-1} \\ &= \tilde{\chi}(\{y \in Q \mid y < x\}) + \sum_{\emptyset \neq \tau \in \Delta((-\infty, x)_P \setminus Q)} (-1)^{|\tau|} \tilde{\chi}(\{y \in Q \mid y < \min \tau\}) \end{aligned}$$

for any  $x \in (P \cup \{\infty\}) \setminus Q$ .

So (i)  $\implies$  (ii) is clear and (ii)  $\implies$  (i) is proved by the (Artinian) induction on  $x$ . ■

By Lemma 6.9 and Lemma 6.10, we have the following

**Proposition 6.11.** *Let  $A$  be a graded ASL over a field  $k$  generated by a Cohen-Macaulay poset  $P$  and  $Q$  a straightening closed poset ideal of  $P$ . Denote the associated graded ring with respect to  $I = QA$  by  $G$ . Then  $a(G) < 0$  if and only if  $\tilde{\chi}((-\infty, x)_P) = 0$  for any  $x \in (P \cup \{\infty\}) \setminus Q$ .*

Denote the Rees algebra with respect to  $I$  by  $R$  in the setting of Proposition 6.11. Then by Fact 2.2, Proposition 6.2, Theorem 6.3 and Theorem 6.8, we see the following

**Theorem 6.12.** *In the above setting,  $R$  is Cohen-Macaulay if and only if  $P \uplus Q$  is Cohen-Macaulay over  $k$ . In particular,  $R$  is Cohen-Macaulay if and only if  $R_{\text{dis}}$  is Cohen-Macaulay.*

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